LECTURE 27 MONOTONE FUNCTIONS AND SECOND DERIVATIVE TEST

Before we go on with monotone functions and their properties, we complete a couple of examples from the last section.

Example 1. For what values of a, m and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Solution. You need to enforce continuity on $[0, 2]$ and differentiability on $(0, 2)$. The points of interest are $x = 0$ and $x = 1$.

Continuity at $x = 0$ would require

$$f(0) = 3 = \lim_{x \to 0^+} f(x) = a.$$

Continuity at $x = 1$ would require

$$f(1) = m + b = \lim_{x \to 1^-} \left(-x^2 + 3x + 3\right) = 5.$$

Differentiability at $x = 1$ would require

$$\begin{aligned} \lim_{x \to 1^-} f'(x) &= \lim_{x \to 1^+} f'(x) \\ \implies -2x + 3 \,|_{x=1} &= m \\ \implies m &= 1 \end{aligned}$$

and therefore $b = 4$.

Example. Find the functions that satisfy $f'(t) = \cos(5t) - 3\sin\left(\frac{t}{5}\right)$.

Solution. Trial and error. You first must have quick sense of what derivative a function gives you, and then think backwards.

Looking at the first term, the function that can possibly give you $\cos(5t)$ as a derivative is something like $\sin(5t)$. However,

$$\frac{d}{dt}\sin(5t) = 5\cos(5t)$$

where you overkilled with the factor. Thus, you correct your original guess with $\frac{1}{5}\sin(5t)$.

Similarly, to get $\sin\left(\frac{t}{5}\right)$ as a derivative, you may guess $-\cos\left(\frac{t}{5}\right)$. But you find

$$\frac{d}{dt} - \cos\left(\frac{t}{5}\right) = \frac{1}{5}\sin\left(\frac{t}{5}\right)$$

which you underestimated by a factor of $\frac{1}{5}$. Therefore, you correct the guess by $-5\cos\left(\frac{t}{5}\right)$.

Lastly, the corollaries tell us to always add C at the end, when two functions have the same derivative. Altogether, we have

$$f(t) = \frac{1}{5}\sin(5t) + 3 \times 5\cos\left(\frac{t}{5}\right) = \frac{1}{5}\sin(5t) + 15\cos\left(\frac{t}{5}\right).$$

First Derivative Test

We did first derivative test already. At a critical point $x = c$,

1. if f' changes from negative to positive at c, then f has a local minimum at $x = c$.
2. if f' changes from positive to negative at c, then f has a local maximum at $x = c$.
3. if f' does not change sign (that is, f' has the same sign on both sides of c), then f has no local extremum at c.

Differentiable Monotone Functions

The first derivative test can be made more precise with the notion of monotone functions. First, we state a property of differentiable monotone functions.

Corollary. *Suppose that f is continuous on $[a,b]$ and differentiable on (a,b).*

- *If $f'(x) > 0$ at each point $x \in (a,b)$, then f is increasing on $[a,b]$.*
- *If $f'(x) < 0$ at each point $x \in (a,b)$, then f is decreasing on $[a,b]$.*

Proof. We prove the first case as the second is analogous. Let x_1 and x_2 be any two points in $[a,b]$ with $x_1 < x_2$. The Mean Value Theorem implies that there is $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Now, by hypothesis, $f'(c) > 0$. Along with $x_2 > x_1$, we find that $f(x_2) - f(x_1) > 0$, for every such pair of points x_1 and x_2 in $[a,b]$. Thus f is increasing on $[a,b]$. □

Example. Find the critical points of $f(x) = x^{1/3}(x-4)$ and identify the open intervals on which f is increasing and on which f is decreasing. Then, determine the extrema.

Solution. We find that the critical points satisfy

$$0 = f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} \implies \frac{4(x-1)}{3x^{2/3}} = 0 \implies x = 1, \quad x = 0.$$

The critical points subdivide the whole domain into three partitions, $(-\infty, 0), (0,1), (1,\infty)$ and we shall find how the function is behaving on them. Since the domain has no endpoints, the critical points are the only places where the function might achieve extrema. So, we look at the sign of $f'(x)$ by either plugging in some values in the respective intervals, or we argue by solving inequalities.

Note that

$$f'(x) = \frac{4(x-1)}{3x^{2/3}}$$

has a positive numerator. Therefore, the sign of f' is completely determined by the sign of $(x-1)$. If $x < 0$, $f'(x) < 0$, which means f is decreasing on $(-\infty, 0)$. At the same time, if $0 < x < 1$, $f'(x)$ is still decreasing (on $(0,1)$). Lastly, for $x > 1$, we find that $f'(x) > 0$ and thus f is increasing on $(1, \infty)$.

By the first derivative test, f has a local minimum at $x = 1$.

Example. On $[0, 2\pi]$, find the critical points of $g(x) = \sin^2(x) - \sin(x) - 1$, identify the open intervals on which f is increasing and on which f is decreasing. Determine the extrema.

Solution. The critical points satisfy

$$0 = g'(x) = 2\sin(x)\cos(x) - \cos(x) \implies \cos(x)(2\sin(x) - 1) = 0$$

which implies

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

So, the intervals we form is

$$\left(0, \frac{\pi}{6}\right), \left(\frac{\pi}{6}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{5\pi}{6}\right), \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right).$$

We find $-, +, -, +, -$ respectively, of the signs of $f'(x)$. Thus, local max at $x = \frac{\pi}{6}, \frac{3\pi}{2}$, local min at $x = \frac{\pi}{2}, \frac{5\pi}{6}$. Now, for absolute extrema, we include the endpoints,

$$g(0) = -1, \quad g(2\pi) = -1$$

while

$$g\left(\frac{\pi}{6}\right) = \frac{1}{4} - \frac{1}{2} - 1 = -\frac{5}{4}, \quad g\left(\frac{\pi}{2}\right) = -1, \quad g\left(\frac{5\pi}{6}\right) = -\frac{5}{4}, \quad g\left(\frac{3\pi}{2}\right) = 1.$$

The absolute maximum is at $g\left(\frac{3\pi}{2}\right) = 1$ and the absolute minimum is at $g\left(\frac{\pi}{6}\right) = -\frac{5}{4}$ and $g\left(\frac{5\pi}{6}\right) = -\frac{5}{4}$.

The Second Dervative and Concavity

There are two ways of increasing, $\Yup$ or $\Ydown$. The difference here is how it curves, or whether the curve faces down or up.

Definition. The graph of a differentiable function $y = f(x)$ is

(1) **concave up/convex** on an open interval I if f' is increasing on I.
(2) **concave down** on an open interval I if f' is decreasing on I.

If f'' exists, then we can use the corollary on monotone functions here.

The second dervative test for concavity.

Let $f(x)$ be a twice-differentiable function on some interval I.

(1) If $f'' > 0$ on I, then the graph of f on I is **concave up/convex**.
(2) If $f'' < 0$ on I, then the graph of f on I is **concave down**.